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ON THE RATIONAL, INTEGRAL INVARIANTS OF NILPOTENT ALGEBRAS.*

By Olive C. Hazlett.

Introduction.

§ 1. Relation to literature. In the study of linear algebras, their invariantive classification is one of the most important problems. Peirce, Study, Scheffers, Hawkes, and others* have classified, over the field C of ordinary complex numbers, linear associative algebras in a small number of units; and associative algebras having less than four units, with a modulus, are easily characterized invariantively by invariants obtainable from the characteristic determinants,† but such invariants are not sufficient for the characterization of quaternary associative algebras with a modulus.‡ In view of Wedderburn's§ generalization of a theorem of Cartan,|| we can go a long way toward characterizing the general linear algebra if we can characterize three special kinds of algebras; namely, matric algebras, division algebras, and nilpotent algebras. Over the field C, there is only a finite number of classes of the first two kinds of algebras of a given order; whereas, over the field C, there is an infinite number of classes of nilpotent algebras of any order greater than two; and hence, we centre our attention on nilpotent algebras.

In this paper, we consider rational, integral invariants of such algebras. Now all invariants obtainable from the characteristic determinants are zero for nilpotent algebras; and furthermore, there are no rational, integral invariants (other than constants) for n-ary nilpotent algebras under the total n-ary linear group. We accordingly consider invariants under the group which leaves unaltered the canonical form. For such rational, integral invariants this paper proves theorems analogous to theorems about invariants of algebraic forms, and in particular proves their finiteness. The fundamental invariants are found for the simpler cases.

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* Written while the author was Alice Freeman Palmer Fellow of Wellesley College, 1915–16.

* B. Peirce, American Journal of Mathematics, vol. 4, p. 97; Study, Goettinger Nachrichten, 1889, p. 237; G. Scheffers, Mathematische Annalen, vol. 39, p. 293; Hawkes, American Journal of Mathematics, vol. 24, p. 87, Transactions, vol. 3, p. 312; Cayley, Collected Papers, pp. 60, 105; Buchheim, Messenger of Mathematics, series 2, vol. 16, p. 62.

† Hazlett, Annals of Mathematics, series 2, vol. 16, p. 1.

‡ Hazlett, Chicago Dissertation, § 1, American Journal of Mathematics, vol. 38.

§ Proceedings of London Math. Society, series 2, vol. 6, p. 109.

|| Annales de Toulouse, séries 1, vol. 12, *B*.

§ 2. **Definitions.** A *linear algebra** over the field F is a set of hypercomplex numbers of the form $X = \sum_{i}^{1,n} x_i e_i$ where the *coördinates* x_i range independently over F, and where the *units* e_i are such that

$$e_i e_j = \sum_{k}^{1,n} \gamma_{ijk} e_k \qquad (i, j = 1, \cdots, n),$$

where (i) the *constants of multiplication* γ_{ijk} are in F, (ii) the sum of two numbers X and Y of the algebra is $X + Y = \sum_{i}^{1,n} (x_i + y_i) e_i$ and (iii) numbers of the algebra combine under addition and multiplication according to the distributive law. Unless explicitly stated, we do not assume the commutative nor the associative law of multiplication. A linear algebra A which is such that, for some integer α, $A^\alpha = 0$, is said to be nilpotent.† If α is the smallest such number, α is called the *index* of the algebra.

The *complex*‡ $C = (x_1, \cdots, x_a)$ is defined as the set of all quantities linearly dependent on $x_1, \cdots, x_a$; and the number of linearly independent elements of the complex is called its *order*. If C_1 and C_2 are two complexes, the complex formed by all quantities linearly dependent on the elements of C_1 and C_2 is called the *sum* of C_1 and C_2 and is denoted by $C_1 + C_2$. If y_1 and y_2 are any elements of C_1 and C_2 respectively, the complex of all elements linearly dependent on those of the form $y_1 y_2$ is called the *product* of C_1 by C_2 and is written $C_1 C_2$. If a complex C_2 is contained in a complex C_1, we write $C_2 \leq C_1$.

Suppose we have given the n-ary algebra $A = (e_1, \cdots, e_n)$ with the constants of multiplication γ_{ijk} and the n-ary algebra $A' = (e_1', \cdots, e_n')$ with constants of multiplication γ_{ijk}'. If there exist a non-singular linear homogeneous transformation of the group L on the units of A such that, for the transformed algebra A', the constants of multiplication are equal to the corresponding γ_{ijk}', then A and A' are said to be *equivalent* with respect to the group L. The totality of all algebras equivalent with respect to a group L are said to constitute a *class* of algebras with respect to that group.

Now we know§ that, if A be a nilpotent algebra (not necessarily associative) with index α, then A is the sum of $\alpha - 1$ non-overlapping complexes $B_1, \cdots, B_{\alpha-1}$, such that

$$B_p B_q \leq \sum_{r=p+q}^{\alpha-1} B_r.$$

* For other definitions, see Dickson, Transactions, vol. 4, p. 26.

† For a statement of the two essentially different definitions of a nilpotent algebra see author's dissertation or Wedderburn, l. c., p. 87. The one used here is that due to Wedderburn.

‡ Wedderburn, l. c., p. 79.

§ Wedderburn, l. c., p. 87.

In fact $B_i \equiv A^i \pmod{A^{i+1}}$ for every i. The orders of the complexes B_i are invariants, though the complexes themselves are not invariantive. Moreover, any algebra which can be put in this form is necessarily nilpotent. Hence if a nilpotent algebra be in the above form and such that $B_1 = (e_1, \cdots, e_{\nu'})$ and in general $B_i = (e_{\nu^{(i-1)}+1}, \cdots, e_{\nu^{(i)}})$ $(i = 2, \cdots, \alpha - 1)$, then it is said to be in *canonical form.** Notice that $\nu^{(i)} = \sum_{j=1}^{i} n^{(j)}$ where $n^{(j)}$ is the order of B_j. If $n^{(i)}$ is the order of B_i $(i = 1, \cdots, \alpha - 1)$, the nilpotent algebra of index α is said to be of *genus* $(\alpha; n', \cdots, n^{(\alpha-1)})$.

Invariants for the general nilpotent algebra.

§ 3. Rational, integral invariants of n-ary nilpotent algebra under total n-ary linear group. Consider those rational, integral functions of the constants of multiplication of the general n-ary linear algebra which are invariant under the total linear group of transformations on the n units. Any such invariant I has certain properties analogous to the isobarism of the rational, integral invariants for algebraic forms.

To show this, we need the notion of "weight." Now if we have the algebra

$$e_i e_j = \sum_{k}^{1,n} \gamma_{ijk} e_k \qquad (i, j = 1, \cdots, n),$$

then, when we multiply e_r by $\lambda \neq 0$, and leave other units unchanged, the γ's are subject to the induced transformation where γ_{ijk} is unaltered when $i, j, k \neq r$, and where γ_{ijr} $(i, j \neq r)$ is multiplied by λ^{-1}, γ_{rir} and γ_{irr} $(i \neq r)$ by λ^0, γ_{rrr} by λ, γ_{rjk} and γ_{jrk} $(j, k \neq r)$ by λ and γ_{rrk} $(k \neq r)$ by λ^2. Hence, for convenience, we say that γ_{ijk} $(i, j, k \neq r)$ is of weight 0 in e_r, γ_{ijr} $(i, j \neq r)$ of weight -1 in e_r, γ_{rir} and γ_{irr} $(i \neq r)$ of weight 0, γ_{rrr} of weight 1, γ_{rjk} and γ_{jrk} $(j, k \neq r)$ of weight 1, and γ_{rrk} $(k \neq r)$ of weight 2 in e_r. And we do this for every r. For every unit e_r, we thus assign to γ_{ijk} a weight with respect to that unit, which is counted up in the following way: the first subscript, i, contributes $+1$ toward this weight if $i = r$, but 0 if $i \neq r$; the second, j, contributes $+1$ if $j = r$, otherwise 0; and the third, k, contributes -1 if $k = r$, otherwise 0. The sum of these three partial weights is called the *weight of γ_{ijk} with respect to e_r.*

If we consider a particular rational, integral invariant I of the kind described above, then I is isobaric in e_1. For, if we multiply e_1 by a scalar $\lambda \neq 0$, then the γ's are multiplied as follows,—γ_{111} is multiplied by λ, the γ_{11i} $(i \neq 1)$ are multiplied by λ^2, the γ_{1j1} and γ_{j11} $(j \neq 1)$ are multiplied by λ^0, the γ_{1jk} and γ_{j1k} $(j, k \neq 1)$ are multiplied by λ, the γ_{lm1} $(l, m \neq 1)$

* Author's dissertation, §§ 3, 5.

are multiplied by λ^{-1} and the remaining γ's are unaltered. But, since I is an invariant, we have

$$I(\gamma'_{ijk}) = F(\lambda)I(\gamma_{ijk})$$

where the γ'_{ijk} are the constants of multiplication for the transformed algebra. Hence our statement. In general we have

Lemma I. *For an n-ary linear algebra, a rational, integral invariant under the total n-ary linear group is isobaric in any particular unit* e_i.

Furthermore, since I must be invariant under the transformation which merely interchanges e_1 and e_l, the weight of I in e_1 must equal its weight in e_l, for every l. This is true only for invariants under the *total* linear group.

In particular, consider I when formed for a nilpotent algebra in canonical form. Some terms drop out, owing to the form of the multiplication table. The terms that remain (if any) cannot be of positive weight in e_n and cannot be of negative weight in e_1, and yet must be of the same weight in e_1 and e_n. Hence the terms that remain must be of weight zero in e_1 and e_n, and thus of weight zero in each unit. But for a nilpotent algebra in canonical form, each γ which is not zero is of positive or zero weight with respect to e_1, those of zero weight being the γ_{ijk} $(i, j, k \neq 1)$; and thus I here contains at most only the γ_{ijk} $(i, j, k \neq 1)$. Similarly I can not contain any γ's which depend on e_n. Then I can not contain any γ's which depend on any unit of B_1 or of $B_{\alpha-1}$. Finally, by induction, I when formed for a nilpotent algebra in canonical form must be independent of all the γ's.

In this we have been considering rational, *integral* functions of the γ's, but the conclusion applies also to *any* rational function of the γ's. For consider the invariant $I = I_1/I_2$ where I_1 and I_2 are rational integral functions of the γ's with no common factor. Then, when we make a linear transformation on the units, the γ's are replaced by linear functions of the original γ's, and hence I_j $(j = 1, 2)$ is replaced by a rational integral function of the same degree as I_j. Thus, from the invariancy of I we infer that of I_1 and I_2. Accordingly we have

Theorem I. *Other than constants, there are no rational functions of the constants of multiplication which, under the total n-ary linear group, have the invariantive property for the general n-ary nilpotent algebra or even for a particular set of nilpotent algebras.* This is a generalization of the known fact that all invariants obtainable from the characteristic determinants reduce to zero for a nilpotent algebra.

Invariants for a nilpotent algebra in canonical form.

§ 4. Some properties of rational, integral invariants of nilpotent algebras of genus $(\alpha; n', \cdots, n^{(\alpha-1)})$. In view of the theorem of the last section, we consider the general nilpotent algebra over the field F, of genus

$(\alpha;\ n',\ \cdots,\ n^{(\alpha-1)})$, in canonical form, and inquire into the properties of those rational, integral functions of the constants of multiplication of such an algebra which are invariant under the group G which leaves unaltered the canonical form. This group G consists of the totality of transformations of the form*

$$e_{i_p}' = \sum_{q=p}^{\alpha-1} \sum_{jq} a_{i_p j_q} e_{j_q}, \tag{1}$$

where e_{j_q} is a unit of B_q, e_{i_p}' a unit of B_p' and the a's are in F.

Lemma II. *A rational, integral invariant I of the general (n-ary) nilpotent algebra in canonical form under the group G is homogeneous in the constants of multiplication.* For, since I is invariant under the general transformation of G, it must in particular be invariant under the transformation $e_j' = \lambda e_j$ $(j = 1, \cdots, n)$ $(\lambda \neq 0)$.

Moreover, Lemma I holds for invariants under this group G. For convenience we shall accordingly borrow the term *gradient*, using it here to denote a homogeneous function of the constants of multiplication which is isobaric in each unit.

Although a rational, integral invariant I under G is isobaric in each unit, its weight in e_1 is not in general equal to its weight in e_j, since I is not in general invariant if we interchange e_1 and e_j. If, however, e_j is one of the units of B_1, then I must be invariant when we interchange e_1 and e_j; and thus we have

Lemma III. *For the general nilpotent algebra in canonical form, if I be of weight w_j in e_j, a unit of B_i, then it is of weight w_j in e_k where e_k is any unit of B_i.*

The invariancy of I is expressed by the formula

$$I(\gamma'_{ijk}) = \phi(a_{lm}) I(\gamma_{ijk}), \tag{2}$$

where the γ_{ijk} are the constants of multiplication of the original algebra, the a_{lm} are the coefficients of the transformation, and the γ'_{ijk} the constants of multiplication of the transformed algebra. Now we know from the theory of invariants for algebraic forms, that the factor by which a rational, integral invariant is multiplied under a linear transformation of the variables is a power of the determinant of the transformation, the exponent being given by a simple formula when we know the number of variables, and the degree and weight of our invariant. We would expect an analogous thing to happen here, and such is the case.

When we subject the units of the algebra $E = (e_1, \cdots, e_n)$ to the non-singular transformation

$$e_i' = \sum_{j}^{1,n} a_{ij} e_j \qquad (i = 1, \cdots, n),$$

* Author's dissertation, § 4.

then, if the γ_{ijk} be the constants of multiplication for E, and the γ'_{ijk} the constants of multiplication for the transformed algebra E',

$$\sum_p \gamma'_{ijp} a_{pq} = \sum_{k,l} a_{ik} a_{jl} \gamma_{klq} \qquad (i, j, q = 1, \cdots, n). \tag{3}$$

If in (3) we keep i and j fixed and let q range from 1 to n, we have n homogeneous equations in n unknowns γ'_{ijp} $(p = 1, \cdots, n)$ which we can solve uniquely.

Suppose that, in particular, we have a nilpotent algebra of genus $(\alpha; n', \cdots, n^{(\alpha-1)})$ in canonical form, and that we subject the units to the most general transformation of the group G which leaves unaltered units not in B_1. Then the units of B_1 are transformed according to

T_1:
$$e_{i_1}' = \sum_{q=1}^{\alpha-1} \sum_{j_q} a_{i_1 j_q} e_{j_q},$$

where the determinant

$$A_1 \equiv |a_{i_1 j_1}| \neq 0.$$

Then from (3) on account of the special form of the transformation we have

$$\gamma'_{i_1 j_1 p} = \sum_{k,l} a_{i_1 k} a_{j_1 l} \gamma_{klp},$$

$$\gamma'_{i_1 jp} = \sum_k a_{i_1 k} \gamma_{kjp} \qquad (j > n'),$$

$$\gamma'_{ij_1 p} = \sum_l a_{j_1 l} \gamma_{ilp} \qquad (i > n'),$$

while all other γ''s are equal to the corresponding γ's.

Now these formulæ show that, if I be a rational, integral invariant under G, of total weight $w^{(1)}$ in the units of B_1, then $I(\gamma'_{ijk})$ is a rational, integral homogeneous function of degree $w^{(1)}$ in the $a_{i_1 k}$. But $I(\gamma_{ijk})$ is independent of the coefficients of the transformation, and hence $\phi(a_{lm})$ for the transformation T, is a rational, integral homogeneous function ϕ_1 of degree $w^{(1)}$ in $a_{i_1 k}$. Thus

$$I(\gamma'_{ijk}) = \phi_1(a_{i_1 l}) I(\gamma_{ijk}).$$

Also

$$I(\gamma_{ijk}) = \phi_1(A_1^{-1} \cdot A_{li_1}) I(\gamma'_{ijk})$$
$$= A_1^{-w^{(1)}} \phi_1(A_{li_1}) I(\gamma'_{ijk}).$$

Combining these two results, we have

$$\phi_1(A_{ji_1}) \phi_1(a_{i_1 j}) = A_1^{w^{(1)}},$$

where both functions on the left are rational and integral in the $a_{i_1 j}$. But A_1 is an irreducible function of its elements, when these are regarded as independent variables, and hence

$$\phi_1(a_{i_1 j}) = k_1 A_1^{r_1}.$$

To find the value of r_1, we observe that $\phi_1(A_{ji_1})$ is of degree $r_1 n^{(1)}(n^{(1)} - 1)$ in the $a_{i_1 j}$, and hence

$$r_1 = \frac{w^{(1)}}{n^{(1)}}.$$

Furthermore $k_1 = 1$ as we can see by applying the identical transformation.

Similarly, if we subject the units to the most general transformation of the group G which leaves unaltered units not in B_2, the units of B_2 undergo the transformation

T_2:
$$e_{i_2}' = \sum_{q=2}^{a-1} \sum_{j_q} a_{i_2 j_q} e_{j_q},$$

where the determinant

$$A_2 \equiv |\, a_{i_2 j_2} \,| \neq 0.$$

Then

$$\gamma_{i_1 j_1 m_2} = \sum_{n_2} \gamma'_{i_1 j_1 n_2} a_{n_2 m_2},$$

$$\gamma_{i_1 j_1 m_p} = \sum_{n_2} \gamma'_{i_1 j_1 n_2} a_{n_2 m_p} + \gamma'_{i_1 j_1 m_p} \qquad (p > 2),$$

$$\gamma'_{i_2 j_2 m} = \sum_{k,\, l} a_{i_2 k} a_{j_2 l} \gamma_{klm},$$

$$\gamma'_{i_2 j_p m} = \sum_{k} a_{i_2 k} \gamma_{k j_p m} \qquad (p \neq 2),$$

$$\gamma'_{i_p j_2 m} = \sum_{l} a_{j_2 l} \gamma_{i_p l m} \qquad (p \neq 2),$$

while the other γ'_{ijk} are equal to the corresponding γ_{ijk}.

The proof for this case runs very much the same as for the first case, the differences being due to the facts that (i) the $\gamma'_{i_1 j_1 m_p}$ $(p > 2)$ are not in general homogeneous in the $a_{n_2 m}$ and hence conceivably $I(\gamma'_{ijk})$ may not be homogeneous in these a's; and (ii) $I(\gamma'_{ijk})$ is not in general integral in the $a_{n_2 m}$. The first difficulty is obviated when we observe that those terms in $a_{n_2 m}$ arising from the $\gamma_{i_1 j_1 m_p}$ $(p > 2)$ must cancel one another, inasmuch as these terms give rise to no terms of the requisite weight in the units of B_2. As for the second difficulty, we notice that the only way $I(\gamma'_{ijk}) \equiv I'(\gamma_{ijk})$ can fail to be integral in the $a_{n_2 m}$ is by having in the denominator A_2 raised to an integral power, thus:

$$I(\gamma'_{ijk}) \equiv I'(\gamma_{ijk}) \equiv \frac{1}{A_2^{\delta^{(2)}}} I''(\gamma_{ijk}),$$

where I'' is integral in the $a_{n_2 m}$, and therefore $A_2^{\delta^{(2)}} \phi_2(a_{i_2 j})$ is rational, integral and homogeneous in the $a_{i_2 j}$. Then, for this case we infer that

$$\phi_2(a_{i_2 j}) \equiv A_2^{\frac{w^{(2)}}{n^{(2)}}}$$

We obtain a similar result for a transformation T_i which leaves unaltered the units of B_j $(j \neq i)$.

But the most general transformation T of the group G is the product of these transformations T_i, and thus we have

Theorem II. *For a nilpotent algebra* $E = \sum_{i=1}^{\alpha-1} B_i$ *of genus* $(\alpha; n', \cdots, n^{(\alpha-1)})$ *in canonical form, let* $I(\gamma_{ijk})$ *be an invariant (under the group* G *which leaves unaltered the canonical form) of total weight* $w^{(i)}$ *in the units of* B_i. *If we subject the units of* E *to the transformation*

$$e_{i_p}' = \sum_{q=p}^{\alpha-1} \sum_{j_q} a_{i_p j_q} e_{j_q}$$

(belonging to G*) whose determinant* $A = \prod_{p=1}^{\alpha-1} A_p \neq 0$, *where*

$$A_p = |a_{i_p j_p}|,$$

then

$$I(\gamma'_{ijk}) \equiv \prod_{p}^{1, \alpha-1} A_p{}^{r_p} I(\gamma_{ijk}),$$

where the $\gamma''s$ *are the constants of multiplication of the transformed algebra, and where*

$$r_p = \frac{w^{(p)}}{n^{(p)}}.$$

§ 5. Finiteness of rational, integral invariants of nilpotent algebras of genus (3; 2, 1) in canonical form. The multiplication table of the general nilpotent algebra of genus (3; 2, 1) in canonical form is

$$e_i e_j = \gamma_{ij3} e_3 \quad (i, j = 1, 2), \qquad e_k e_3 = 0 = e_3 e_k \quad (k = 1, 2, 3),$$

and here the transformations of the group G are of the form

T:

$$e_i' = \sum_{j}^{1, 3} a_{ij} e_j \qquad (i = 1, 2),$$

$$e_3' = a_{33} e_3 \qquad a_{33}(a_{11}a_{22} - a_{12}a_{21}) \neq 0.$$

By the transformation T, the γ's are subjected to the induced transformation

$$\gamma'_{ij3} = \frac{\sum_{k,l} a_{ik} a_{jl} \gamma_{kl3}}{a_{33}} \equiv f_{ij}(\gamma_{kl3}) \qquad (i, j = 1, 2).$$

And thus, by Lie's theory of continuous groups, if a rational, integral function I of the γ's is invariant under T, I must be annihilated by the two operators

$$U_{12} \equiv (\gamma_{123} + \gamma_{213}) \frac{\partial}{\partial \gamma_{113}} + \gamma_{223} \left(\frac{\partial}{\partial \gamma_{123}} + \frac{\partial}{\partial \gamma_{213}} \right),$$

$$U_{21} \equiv (\gamma_{123} + \gamma_{213}) \frac{\partial}{\partial \gamma_{223}} + \gamma_{113} \left(\frac{\partial}{\partial \gamma_{123}} + \frac{\partial}{\partial \gamma_{213}} \right),$$

and all gradients so annihilated are invariants.* The theory of § 4 tells us that there are no absolute rational, integral invariants.

Now is there only a finite number of rational, integral invariants in the sense that all such invariants are expressible as polynomials in a finite number of them? By analogy with the theory of the Aronhold operators in the study of invariants of algebraic forms, we answer this question by considering the commutator

$$\begin{aligned} (U_{12}, U_{21}) &\equiv U_{12}U_{21} - U_{21}U_{12} \\ &\equiv 2\left(\gamma_{223} \frac{\partial}{\partial \gamma_{223}} - \gamma_{113} \frac{\partial}{\partial \gamma_{113}} \right) \\ &\equiv 2\left\{ \Sigma \gamma_{ij3} \frac{\partial}{\partial \gamma_{ij3}} - \left[2\gamma_{113} \frac{\partial}{\partial \gamma_{113}} + \gamma_{123} \frac{\partial}{\partial \gamma_{123}} + \gamma_{213} \frac{\partial}{\partial \gamma_{213}} \right] \right\}. \end{aligned} \tag{4}$$

If now, in particular, this operator acts on a gradient $\mathfrak{G}$ of degree d, we have

$$(U_{12}, U_{21})\mathfrak{G} \equiv 2d\mathfrak{G} - 2\Sigma(2r_{11} + r_{12} + r_{21})\mathfrak{G}$$

where r_{ij} is the degree of any particular term in γ_{ij3}. But $2r_{11} + r_{12} + r_{21}$ is constant for any gradient, since it equals w_1 the weight in e_1. Thus

$$(U_{12}, U_{21})\mathfrak{G} \equiv 2[d - w_1]\mathfrak{G} \tag{5}$$

and also

$$(U_{12}, U_{21})\mathfrak{G} \equiv -2[d - w_2]\mathfrak{G}. \tag{5'}$$

If $w_1 = w_2$ (as is necessarily the case for an invariant), then

$$(U_{12}, U_{21})\mathfrak{G} = 0,$$

which checks with the fact that an invariant is annihilated by U_{12} and U_{21}.

In particular, if $\mathfrak{G}$ is such that $U_{12}\mathfrak{G} = 0$, then from (5) we have

$$U_{12}U_{21}^{p}\mathfrak{G} = 2\left[p(d - w_1) - \sum_{i=1}^{p-1} i \right] U_{21}^{p-1}\mathfrak{G} \quad (p = 1, 2, \cdots). \tag{6}$$

But $U_{21}\mathfrak{G}$ is a gradient whose weight in e_2 is one less than the corresponding

* The proof of this proceeds exactly as the proof of the analogous theorem in the theory of algebraic invariants, as in Elliott, Algebra of Quantics, 2d ed., p. 113. It also follows from the Lie theory, as in Lie, Transformationsgruppen, p. 215, or Cohen, Lie Theory of One-Parameter Groups.

weight of $\mathfrak{G}$, and accordingly there is some positive integer r such that $U_{21}^r\mathfrak{G} = 0$. Hence we have

$$0 = 2\left[r(d - w_1) - \sum_{i=1}^{r-1} i \right] U_{21}^{r-1}\mathfrak{G}. \tag{6'}$$

If $d < w_1$, then $p(d - w_1) - \sum_{i=1}^{p-1} i < 0$ for every p and therefore (6') shows that $U_{21}^{r-1}\mathfrak{G} \equiv 0$ which, in view of (6), implies that $\mathfrak{G} \equiv 0$. Thus we have

Theorem III. *For genus* (3; 2, 1), *if* $\mathfrak{G}$ *is a gradient of degree* d *and of weights* w_1 *and* w_2 *in* e_1 *and* e_2 *respectively, which is annihilated by* U_{12}, *then* $d - w_1 \geq 0$, *which is equivalent to* $w_2 - w_1 \geq 0$. *Similarly, if* $\mathfrak{G}$ *is annihilated by* U_{21}, *then* $d - w_2 \geq 0$ *which is equivalent to* $w_1 - w_2 \geq 0$.

From this theorem we have immediately

Theorem IV. *For genus* (3; 2, 1) *a gradient* $\mathfrak{G}$ *of degree* d *and weights* w_1 *and* w_2 *in* e_1 *and* e_2 *respectively, which is such that* $w_1 = w_2$ *and which is annihilated by* U_{12} (*or by* U_{21}), *is an invariant under the group* G.

For if $U_{12}\mathfrak{G} \equiv 0$, then from (5) we have at once $U_{12}U_{21}\mathfrak{G} \equiv 0$; and hence either $U_{21}\mathfrak{G} \equiv 0$ or $U_{21}\mathfrak{G}$ is a gradient of weights w_1' and w_2' such that $w_1' > w_2'$.

If $\mathfrak{G}$ is a gradient of degree d and weights w_1 and w_2, then (5) gives at once

$$(U_{12}U_{21}^2 - U_{21}^2U_{12})\mathfrak{G} \equiv 2[2(d - w_1) - 1]U_{21}\mathfrak{G};$$

and in general, for every positive integer r,

$$(U_{12}U_{21}^r - U_{21}^rU_{12})\mathfrak{G} \equiv 2\left[r(d - w_1) - \sum_{i=1}^{r-1} i \right] U_{21}^{r-1}\mathfrak{G}. \tag{7}$$

Furthermore

$$U_{12}^rU_{21}^r\mathfrak{G} \equiv U_{12}U_{21}\prod_{j=2}^{r}\left\{ U_{12}U_{21} + 2\left[(j-1)(d - w_1) - \sum_{i=1}^{j-1} i \right] \right\}\mathfrak{G}.$$

If $\mathfrak{G}$ be of weight w_2 in e_2, then $U_{21}^{w_2+1}\mathfrak{G} \equiv 0$, and accordingly $U_{12}U_{21}$ annihilates

$$\prod_{j=1}^{w_2}\left\{ U_{12}U_{21} + 2\left[j(d - w_1) - \sum_{i=1}^{j} i \right] \right\}\mathfrak{G}. \tag{8}$$

If, in particular, $\mathfrak{G}$ is such that $w_1 = w_2 = d$, then (8) is a gradient whose weight in e_1 equals its weight in e_2, and therefore the result of operating U_{21} on (8) is identically zero or is a gradient whose weight in e_1 is greater than its weight in e_2. Hence, by Theorem III, U_{21} annihilates expression (8); and consequently, by Theorem IV, we have

Theorem V. *The operator*

$$\prod_{j=1}^{\infty}\left\{ \frac{U_{12}U_{21}}{2\sum_{i=1}^{j} i} - 1 \right\}$$

when applied to any gradient of a nilpotent algebra of genus (3; 2, 1) *in canonical form, for which* $w_1 = w_2$, *produces a function which is invariant under the group G. When applied to an invariant under G, this operator produces the invariant itself.*

We are now in a position to show that, for a nilpotent algebra of genus (3; 2, 1) in canonical form, all rational integral invariants under the group G are expressible rationally and integrally in a finite number of such invariants. For by Hilbert's Theorem about an infinite sequence of polynomials,* any such rational integral invariant I is expressible in the form

$$I \equiv \sum_{j}^{1,m} C_j I_j,$$

where m is a finite number, and where the I_j are rational integral invariants, the same for every invariant I. Since I and the I_j are gradients for which $w_1 = w_2$, we may suppose that the C_j are also. Applying Theorem V, we have

$$I \equiv \sum_{j}^{1,m} I_j' I_j,$$

where the I_j' are invariants for genus (3; 2, 1) in canonical form. Then by induction we have

Theorem VI. *For nilpotent algebras of genus* (3; 2, 1) *in canonical form, the rational integral invariants under the group G leaving unaltered the canonical form are polynomials in a finite number of such invariants, called the fundamental invariants.*

§ 6. **Fundamental rational, integral invariants for genus (3; 2, 1).** Since any rational, integral invariant under the group G must be annihilated by the operators U_{12} and U_{21} of § 5, it follows immediately that there is essentially only one linear invariant here, namely $\gamma_{123} - \gamma_{213}$, which vanishes for commutative algebras and no others. All quadratic invariants are linear combinations of the square of the linear invariant and $\gamma_{113}\gamma_{223} - \gamma_{123}\gamma_{213}$, the determinant of the (covariant)† bilinear form. As the only cubic invariants are functions of these, we would suspect that these are the fundamental invariants.‡

By the theory of rational, integral invariants for algebraic forms, we know that the determinant $D \equiv |\gamma_{ij3}|$ $(i, j = 1, 2)$ is the only invariant for the quadratic form $\sum_{i,j}^{1,2} \gamma_{ij3} x_i x_j$ $(\gamma_{ij3} = \gamma_{ji3})$ in the sense that all other such

* "Ueber die Theorie der algebraischen Formen," Mathematische Annalen, vol. 36, p. 474.

† Author's dissertation, § 11.

‡ That is, for the field of rational numbers, for the field of reals, or for the field of ordinary complex numbers.

invariants are expressible as polynomials in it. But this quadratic form is a covariant,* under the group G, of the nilpotent algebra of genus (3; 2, 1) in canonical form; and thus, translating our knowledge of D over to the invariants for the linear algebra, we see that, when $\gamma_{ij3} = \gamma_{ji3}$, the only rational integral invariants of the general nilpotent algebra of genus (3; 2, 1) in canonical form become polynomials in $\gamma_{113}\gamma_{223} - \gamma_{123}\gamma_{213}$. In other words, every rational integral invariant for genus (3; 2, 1) is of the form

$$aD^p + (\gamma_{123} - \gamma_{213})\phi, \tag{9}$$

where ϕ is homogeneous and of degree $2p - 1$ in the γ's. But $\gamma_{123} - \gamma_{213}$ is an invariant for this genus and hence so also is ϕ. Thus by induction we have

Theorem VII. *Under the group G, every rational integral invariant of the nilpotent algebra of genus* (3; 2, 1) *in canonical form is a polynomial in* $\gamma_{113}\gamma_{223} - \gamma_{123}\gamma_{213}$ *and* $\gamma_{123} - \gamma_{213}$.

§ 7. **Finiteness of rational, integral invariants for genus (3; 3, 1).** For genus (3; 3, 1) the canonical form is

$$e_ie_j = \gamma_{ij4}e_4 \quad (i, j = 1, 2, 3), \qquad e_4e_k = 0 = e_ke_4 \quad (k = 1, \cdots, 4),$$

the transformations of the group G are of the form

$$e_i' = \sum_j^{1,4} a_{ij}e_j \qquad (i = 1, 2, 3),$$

$$e_4' = a_{44}e_4, \qquad a_{44}|a_{ij}| \neq 0 \qquad (i, j = 1, 2, 3),$$

and the constants of multiplication are subject to the induced transformation

$$\gamma'_{ij4} = \frac{1}{a_{44}} \sum_{k,l}^{1,3} a_{ik}a_{jl}\gamma_{kl4} \equiv f_{ij}(\gamma_{kl4}) \qquad (i, j = 1, 2, 3).$$

Here the annihilators are

$$U_{12} \equiv (\gamma_{124} + \gamma_{214})\frac{\partial}{\partial\gamma_{114}} + \gamma_{224}\frac{\partial}{\partial\gamma_{124}} + \gamma_{234}\frac{\partial}{\partial\gamma_{134}} + \gamma_{224}\frac{\partial}{\partial\gamma_{214}} + \gamma_{324}\frac{\partial}{\partial\gamma_{314}},$$

$$U_{13} \equiv (\gamma_{134} + \gamma_{314})\frac{\partial}{\partial\gamma_{114}} + \gamma_{324}\frac{\partial}{\partial\gamma_{124}} + \gamma_{334}\frac{\partial}{\partial\gamma_{134}} + \gamma_{234}\frac{\partial}{\partial\gamma_{214}} + \gamma_{334}\frac{\partial}{\partial\gamma_{314}},$$

together with the operators U_{21}, U_{23}, U_{31} and U_{32} obtained from these two by a permutation of the subscripts 1, 2, 3.

By § 4, there are no linear nor quadratic invariants here; in fact an invariant must be of degree three or a multiple of three. All cubic invariants are linear combinations of

$$|\gamma_{ij4}| \qquad (i, j = 1, 2, 3), \tag{10}$$

* Author's dissertation, § 11.

the discriminant of the (covariant) bilinear form $\sum_{i,j}^{1,3} \gamma_{ij4} x_i y_j$, and

$$\Sigma\gamma_{114}(\gamma_{234} - \gamma_{324})^2 + \Sigma\gamma_{124}\gamma_{314}(\gamma_{234} - \gamma_{324}), \tag{10'}$$

which vanishes for commutative algebras. By analogy with genus (3; 2, 1) we would surmise that these are the fundamental invariants. To show that such is the case, we will base our work on our knowledge of genus (3; 2, 1).

By the same kind of argument as that used in § 6, we know that any invariant for genus (3; 3, 1) is of the form

$$a|\gamma_{ij4}|^p + \chi(\gamma_{kl4}), \qquad i, j = 1, 2, 3, \tag{11}$$

where χ vanishes when $\gamma_{ij4} - \gamma_{ji4} = 0$ $(i, j = 1, 2, 3)$. But any invariant for the general algebra under the total group G is invariant under the group of transformations

$$\begin{aligned} e_i' &= \sum_{j=1}^{2} a_{ij} e_j \qquad (i = 1, 2), \\ e_3' &= \sum_{j=1}^{3} a_{3j} e_j, \\ e_4' &= a_{44} e_4, \end{aligned} \tag{12}$$

where

$$a_{33}a_{44}(a_{11}a_{22} - a_{12}a_{21}) \neq 0.$$

Hence, if

$$\gamma_{134} - \gamma_{314} = 0 = \gamma_{234} - \gamma_{324}, \tag{13}$$

any invariant for genus (3; 3, 1) has the invariantive property for (e_1, e_2, e_4) —a general algebra of genus (3; 2, 1)—after γ'_{134}, γ'_{234} and γ'_{334} are replaced by their expressions in terms of the γ_{ij4}, no matter what be the values of γ_{134}, γ_{234} and γ_{334}.

For an algebra of genus (3; 3, 1) where (13) is satisfied, any invariant (11) becomes

$$a|\gamma_{ij4}|^p + \chi'(\gamma_{kl4}), \qquad i, j = 1, 2, 3, \tag{11'}$$

where χ' vanishes when $\gamma_{124} = \gamma_{214}$. Now in χ' there are no terms in γ_{134} $(= \gamma_{314})$ or γ_{234} $(= \gamma_{324})$. This can be shown quickly by considering the parastrophic algebra, or by the following elementary proof by induction.

Suppose that in χ', of the terms containing γ_{334} to a power greater than h, there are none actually containing γ_{134} or γ_{234}; and consider that term which contains γ_{134} to the kth power and no higher,

$$\gamma_{334}^h \gamma_{134}^k \gamma_{234}^l \psi, \tag{14}$$

where ψ is independent of γ_{134} $(= \gamma_{314})$ and γ_{234} $(= \gamma_{324})$. By Lemma 1, there is only one such term.

If we subject the algebra to the transformation

$$e_i' = e_i \qquad (i \neq 3),$$
$$e_3' = e_1 + e_3,$$

the γ's are subjected to the transformation

$$\gamma'_{ij4} = \gamma_{ij4} \qquad (i, j = 1, 2),$$
$$\gamma'_{134} = \gamma_{114} + \gamma_{134},$$
$$\gamma'_{314} = \gamma_{114} + \gamma_{314},$$
$$\gamma'_{234} = \gamma_{214} + \gamma_{234},$$
$$\gamma'_{324} = \gamma_{124} + \gamma_{324},$$
$$\gamma'_{334} = \gamma_{114} + \gamma_{134} + \gamma_{314} + \gamma_{334}.$$

Since ψ is of weight zero in e_3, (14) for $(k, l) \neq (0, 0)$ is replaced by

$$\gamma^h_{334}[\gamma^k_{134}\gamma^l_{234} + \gamma^k_{114}\gamma^l_{214} + \alpha]\psi + \beta\psi,$$

where α contains only terms whose weight in e_1 is less than $2k + l$, the weight of $\gamma^k_{114}\gamma^l_{214}$, and where β does not involve γ_{334} to a power as high as h. Since χ' is unaltered under this transformation for all values of γ_{334}, γ_{134} $(= \gamma_{314})$ and γ_{234} $(= \gamma_{324})$ after the γ'_{ijk} have been replaced by their expressions in terms of the γ_{ijk}, then in χ' there must be a term to cancel $\gamma^h_{334}\gamma^k_{114}\gamma^l_{214}\psi$, if $\psi \neq 0$, $(k, l) \neq (0, 0)$. But the only terms of χ' (γ'_{ijk}) which will contribute terms containing γ^h_{334} are those containing γ'^g_{334} $(g \geq h)$. By the hypothesis about h and Lemma 1, $g = p$. But $\gamma'^p_{334}\psi_p$ contributes only the term $\gamma^h_{334}\gamma^{p-h}_{114}\psi_p$ which might cancel $\gamma^h_{334}\gamma^k_{114}\gamma^l_{214}\psi$. Thus $p - h \leq k$. Also $2p = 2h + k + l$, since χ' is isobaric in e_3; and hence $k \geq l$. Similarly $l \geq k$; and accordingly $k = l$.

Now apply to the algebra the transformation

$$e_1' = e_1 + e_2, \tag{15}$$
$$e_i' = e_i \qquad (i \neq 1).$$

Then, since χ' is absolutely unaltered under (15), any term of the form

$$\gamma'^h_{334}\gamma'^k_{134}\gamma'^k_{234}\psi(\gamma')$$

for $k > 0$ contributes the term

$$\gamma^h_{334}\gamma^{2k}_{234}\psi(\gamma)$$

which must be cancelled. Hence, since there is no term which could possibly cancel this when $k > 0$, $\psi \not\equiv 0$, we must have finally that $k = l = 0$, if $\psi \not\equiv 0$.

Accordingly the invariant χ, of degree $3p$, when (13) is satisfied becomes the product of γ_{334}^p by a function of degree $2p$ which is invariant for the subalgebra (e_1, e_2, e_4) of genus $(3; 2, 1)$. Hence, by Theorem VII, we deduce

Lemma IV. *For nilpotent algebras of genus* $(3; 3, 1)$ *in canonical form, any rational integral invariant under the group G is necessarily of the form*

$$I = a\,|\,\gamma_{ij4}\,|^{\,p} + \chi(\gamma_{kl4})$$

where

$$\chi \equiv \sum_3 \left(\sum_{q=1}^{p} a_q \gamma_{334}^p (\gamma_{124} - \gamma_{214})^{2q} (\gamma_{114}\gamma_{224} - \gamma_{124}\gamma_{214})^{p-q} \right)^* + \theta,$$

where θ *is rational, integral and homogeneous in the* γ*'s and vanishes when any one of the three conditions*

$$\gamma_{124} - \gamma_{214} = 0 = \gamma_{134} - \gamma_{314}$$

$$\gamma_{214} - \gamma_{124} = 0 = \gamma_{234} - \gamma_{324}$$

$$\gamma_{314} - \gamma_{134} = 0 = \gamma_{324} - \gamma_{234}$$

is satisfied.

Applying this lemma to the case where $w^{(1)} = 6$, we have

$$\chi = b \sum_3 \gamma_{114}(\gamma_{234} - \gamma_{324})^2 + c \sum_6 \gamma_{124}\gamma_{234}(\gamma_{134} - \gamma_{314}).$$

Since χ must be invariant under the transformation

$$e_1' = e_1 + xe_2,$$

$$e_i' = e_i \qquad (i \neq 1),$$

where x is arbitrary, we have $c = -b$, which agrees with the first part of this section.

In any invariant of the nilpotent algebra of genus $(3; 3, 1)$, we can kill off the terms of the form

$$\Sigma\gamma_{334}^p(\gamma_{124} - \gamma_{214})^{2q}(\gamma_{114}\gamma_{224} - \gamma_{124}\gamma_{214})^{p-q}$$

one by one by means of

$$|\,\gamma_{ij4}\,|^{\,p-q}[\Sigma\gamma_{334}(\gamma_{124} - \gamma_{214})^2 - \Sigma\gamma_{124}\gamma_{234}(\gamma_{134} - \gamma_{314})]^q$$

and thus we obtain an invariant K which vanishes when any one of the three conditions

$$\gamma_{124} - \gamma_{214} = 0 = \gamma_{134} - \gamma_{314},$$

(16)

$$\gamma_{214} - \gamma_{124} = 0 = \gamma_{234} - \gamma_{324},$$

$$\gamma_{314} - \gamma_{134} = 0 = \gamma_{324} - \gamma_{234}$$

is satisfied.

* The first summation sign extends over all permutations of the indices 1, 2, 3 which produce distinct terms of the type indicated.

Now

$$K \equiv \sum_{r,s,t} (\gamma_{124} - \gamma_{214})^r (\gamma_{234} - \gamma_{324})^s (\gamma_{134} - \gamma_{314})^t K_{r,s,t}$$

where each $K_{r,s,t}$ is independent of γ_{124}, γ_{234} and γ_{134} and where $K_{r,0,0} \equiv K_{0,s,0} \equiv K_{0,0,t} \equiv 0$ for every r, s, t, since K vanishes whenever any one of the conditions (16) is satisfied. Consider the set of terms where r has the least value r_0, such that some $K_{r_0,s,t} \not\equiv 0$, and $s + t = q$ has a fixed value. Of these, consider that term where s has the greatest value, s_0. Then the corresponding $K_{r,s,t}$ has its weight in e_2 less than that of any other term of this set.

When we subject the algebra to the transformation (15), then the term

$$(\gamma_{124} - \gamma_{214})^{r_0} (\gamma_{234} - \gamma_{324})^{s_0} (\gamma_{134} - \gamma_{314})^{t_0} K_{r_0,s_0,t_0} \qquad (s_0 + t_0 = q),$$

if $t_0 \neq 0$, is replaced by

$$(\gamma_{124} - \gamma_{214})^{r_0} [(\gamma_{234} - \gamma_{324})^{s_0} (\gamma_{134} - \gamma_{314})^{t_0} K_{r_0,s_0,t_0} + (\gamma_{234} - \gamma_{324})^q K_{r_0,s_0,t_0} + \alpha],$$

where no term of α can cancel $(\gamma_{234} - \gamma_{324})^q K_{r_0,s_0,t_0}$. Since K is invariant under (15), the term $(\gamma_{124} - \gamma_{214})^{r_0} (\gamma_{234} - \gamma_{324})^q K_{r_0,s_0,t_0}$ must be cancelled if $K_{r_0,s_0,t_0} \not\equiv 0$, $t_0 \neq 0$. But there is no term in $(\gamma_{124} - \gamma_{214})^{t_0} (\gamma_{234} - \gamma_{324})^q$ which has its weight in e_2 small enough to do this, on account of the way s_0 was chosen; and accordingly we must have $K_{r_0,s_0,t_0} \equiv 0$ if $t_0 \neq 0$.

Thus $\gamma_{134} - \gamma_{314}$ is not a factor of K. But $(\gamma_{124} - \gamma_{214})^{r_0}$ is a factor of K, and accordingly $r_0 = 0$, in view of the invariancy of K under the transformation

$$e_i' = e_i \qquad (i \neq 2, 3),$$

$$e_2' = e_3,$$

$$e_3' = e_2.$$

But by previous remarks, $K_{0,s_0,0} \equiv 0$; and consequently $K_{r_0,s_0,t} \equiv 0$ for every t. Hence, in view of the definition of r_0 and s_0, $K \equiv 0$; and we have

Theorem VIII. *Any rational integral invariant of the nilpotent algebra of genus* (3; 3, 1) *in canonical form is a polynomial in the invariants*

$$\left| \gamma_{ij4} \right|_{i,j=1,2,3} \quad \text{and} \quad \sum_3 \gamma_{334} (\gamma_{124} - \gamma_{214})^2 - \sum_6 \gamma_{124} \gamma_{234} (\gamma_{134} - \gamma_{314}).$$

§ 8. Finiteness proof for genus $(\alpha;\ n', \cdots, n^{(\alpha-1)})$. To prove the finiteness of the rational, integral invariants for the general case, we make use of a theorem which is essentially one proved by Gordan,* F. Mertens,† and Hilbert,‡ and which for $n' = 3$, $n'' = 2$ runs as follows: Let

* "Invariantentheorie," vol. II, § 9.

† Sitzungsberichte der Kais. Akad. der Wiss. zu Wien, vol. 95, p. 942.

‡ L. c., p. 524.

$E = (e_1, e_2, e_3, e_4, e_5)$ be a nilpotent algebra of genus (3; 3, 2) in canonical form where the units are subjected to the transformation

$$(17)\qquad \begin{aligned} e_i' &= \sum_{j}^{1,5} a_{ij}e_j \qquad (i = 1, 2, 3), \\ e_k' &= \sum_{l}^{4,5} b_{kl}e_l \qquad (k = 4, 5) \end{aligned}$$

with

$$A_1 \equiv \left| a_{ij} \right|_{i,j=1,2,3} \neq 0, \qquad A_2 \equiv \left| b_{kl} \right|_{k,l=4,5} \neq 0.$$

Let γ_{ijk} and γ'_{ijk} be the constants of multiplication for the original algebra E and the transformed algebra respectively. Finally, let $H(\gamma_{ijk})$ be a rational, integral function of the coefficients of multiplication of E, and let

$$\Omega_a = \sum_{i,j,k} (\pm\, ijk) \frac{\partial^3}{\partial a_{1i}\partial a_{2j}\partial a_{3k}},$$

where the summation extends over all permutations of 1, 2 and 3 and where $(\pm\, ijk)$ denotes $+1$ or -1 according as i, j, k is an even or odd permutation; similarly let

$$\Omega_b \equiv \sum_{f,g} (\pm\, fg) \frac{\partial^2}{\partial b_{4f}\partial b_{5g}},$$

where the summation extends over all permutations of 4 and 5 and where $(\pm\, fg)$ is $+1$ or -1 according as f, g is an odd or an even permutation. Then, if p_1 and p_2 are such that

$$J(\gamma) \equiv \Omega_a^{p_1}\, \Omega_b^{p_2}\, A_1^{q_1}\, A_2^{q_2}\, H(\gamma')$$

is free of the coefficients of the transformation, $J(\gamma)$ is an invariant of the algebra. The proof runs exactly as in Hilbert's article.

Now, by the theorem of Hilbert quoted in § 5, any rational, integral invariant I of the nilpotent algebras of genus (3; 3, 2) in canonical form may be written

$$I(\gamma) \equiv \sum_{j}^{1,m} C_j(\gamma) I_j(\gamma)$$

where m is a finite number, the I_j are rational, integral invariants for genus (3; 3, 2) the same for every invariant I, and where the C_j are isobaric, rational, integral homogeneous functions of the constants of multiplication. Furthermore, we may take the $C_j(\gamma)$ such that the combined weight of C_j and I_j in B_1 is the weight of I in B_1, and that their combined weight in B_2 is the weight of I in B_2.

When we subject the units to the transformation (17), this equation becomes

$$(18)\qquad A_1^{\frac{w^{(1)}}{3}} A_2^{\frac{w^{(2)}}{2}} I(\gamma) \equiv \sum_{j}^{1,m} A_1^{\frac{w_j^{(1)}}{3}} A_2^{\frac{w_j^{(2)}}{2}} C_j(\gamma') I_j(\gamma)$$

where $w^{(1)}$ and $w^{(2)}$ are the total weights of I in the units of B_1 and B_2 respectively, and where $w_j^{(1)}$ and $w_j^{(2)}$ are the total weights of I_j in the units of B_1 and B_2 respectively. Moreover, on account of the way the C_j were chosen, both sides are of the same degree in the a's and in the b's.

Now, unlike the proof of the analogous theorem for invariants of algebraic forms, we can not apply the theorem of Hilbert stated at the beginning of this section directly to (18). For, since $w^{(2)}$ is negative, we can not apply Ω_b to (18) a number of times sufficient to free both sides of the equation from the coefficients of the transformation (17). But, by (3) § 4, each γ' is a rational, integral function of the coefficients of the transformation divided by A_2. If, accordingly, we multiply both sides of (18) by A_2^p, where p is chosen sufficiently large, we have an equation between two functions which are rational, integral and of the same degree in the a's and in the b's.

If we now operate on the resulting equation with $\Omega_a^{w^{(1)}/3}\Omega_b^{[w^{(2)}/2]+p}$ we have, by Hilbert's theorem,

$$NI(\gamma) \equiv \sum_j^{1,\,m} I_j'(\gamma) I_j(\gamma),$$

where the I_j' are rational integral invariants, and where N is a constant different from zero. Thus, by induction, we prove the finiteness of the rational, integral invariants for genus (3; 3, 2).

In a similar manner we prove

Theorem IX. *For the nilpotent algebra of genus* $(\alpha;\ n',\ \cdots,\ n^{(\alpha-1)})$ *in canonical form, the rational, integral invariants are expressible as polynomials in a finite number of such invariants.*

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